1. (i) (a) $f(x) = x^3 - 3x^2 + 4$

$(x-2)$ is a factor of $f(x) \Leftrightarrow f(2) = 0$

$f(2) = 8 - 12 + 4 = 0$ therefore $(x-2)$ is a factor. *(M1)*

(b) **Method 1**

$$\begin{array}{r|llll} & x^2 & -x & & -2 \\ \hline (x-2) & x^3 & -3x^2 & & +4 \\ & x^3 & -2x^2 & & \\ \hline & & -x^2 & & \\ & & -x^2 & +2x & \\ \hline & & & -2x & +4 \\ & & & -2x & +4 \\ \hline & & & & 0 \end{array}$$

Method 2 By synthetic division

$$\begin{array}{c|cccc} 2 & 1 & -3 & 0 & 4 \\ & & 2 & -2 & -4 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

Therefore $f(x) = (x-2)(x^2 - x - 2)$

$f(x) = (x-2)^2(x+1)$ *(M1)(A1)*

Full marks can b obtained for parts (a) and (b) b using eithe method shown i part (b)

(c) $\dfrac{3}{x^3 - 3x^2 + 4} = \dfrac{3}{(x+1)(x-2)^2}$

Let $\dfrac{3}{x^3 - 3x^2 + 4} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$ *(M1)*

Therefore $3 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

put $x = 2$, $3 = 3C$ therefore $C = 1$

put $x = -1$, $3 = 9A$ therefore $A = \dfrac{1}{3}$

put $x = 0$, $3 = \dfrac{4}{3} - 2B + 1$ therefore $B = -\dfrac{1}{3}$ *(M1)(A1)*

Therefore $\dfrac{3}{x^3 - 3x^2 + 4} = \dfrac{1}{3(x+1)} - \dfrac{1}{3(x-2)} + \dfrac{1}{(x-2)^2}$ *(A1)*

continued . . .

Question 1 continued

(d) $\int \frac{3}{x^3 - 3x^2 + 4} dx = \int \frac{1}{3(x+1)} - \frac{1}{3(x+2)} + \frac{1}{(x-2)^2} dx$

$= \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x-2| - \frac{1}{(x-2)} + C$ *(M1)(A1)* Deduct 1 mark for each error

(ii) (a) For intersection with x-axis put $y = 0$

Therefore $\frac{x^2}{a^2} = 1$

$\Rightarrow x = \pm a$

The coordinates of the points of intersection are $(-a, 0)$ and $(a, 0)$ *(M1)(A1)*

(b) Rearranging: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Leftrightarrow \quad b^2x^2 + a^2y^2 = a^2b^2$

$\Leftrightarrow \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ *(A1)*

The volume, V, is given by:

$$V = \pi\int_{-a}^{a} y^2\, dx$$

$$V = \pi\int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2)dx$$

$$= \frac{\pi b^2}{a^2}\left[a^2x - \frac{1}{3}x^3\right]_{-a}^{a}$$ *(M1)(A1)*

$$= \frac{\pi b^2}{a^2}\left[a^3 - \frac{1}{3}a^3 - \left(-a^3 + \frac{1}{3}a^3\right)\right]$$

$$V = \frac{4\pi ab^2}{3}$$ *(M1)(A1)*

(c) If $b = a$, then the curve is $x^2 + y^2 = a^2$ which is a circle centre $(0, 0)$, radius a. Therefore, the solid is a sphere, centre $(0, 0)$, radius a. *(A2)* Full marks awarded if b is used in place of a

2. (i) (a)

$$\begin{aligned} \frac{17}{6} &= \frac{4(1+t^4)}{8+t^4} \\ 136+17t^4 &= 24+24t^4 \\ 112 &= 7t^4 \\ 16 &= t^4 \\ t &= 2 \end{aligned}$$

The negative value of t can be ignored since $t \geq 0$ *(M1)(A1)*

(b) The rate of change of the radius is $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{(8+t^4)\times 16t^3 - 16t^3 \times (1+t^4)}{(8+t^4)^2} \\ &= \frac{112t^3}{(8+t^4)^2} \text{ cm min}^{-1} \end{aligned}$$

(M1)(A2)

(c) The rate of change of area is $\frac{\mathrm{d}A}{\mathrm{d}t}$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} \qquad A = \pi r^2 \text{ therefore } \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$$

Therefore $\frac{\mathrm{d}A}{\mathrm{d}t} = 2\pi r \times \frac{112t^3}{(8+t^4)^2}$ *(M1)(A1)*

when $r = \frac{17}{6}$ cm, $t = 2$ minutes

Therefore $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{238}{27}\pi \text{ cm}^2\text{ min}^{-1}$ or $27.7 \text{cm}^2 \text{ min}^{-1}$ *(A1)*

continued . . .

Question 2 continued

(d) To find a point of inflection put $\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=0$ and test the value of $\frac{\mathrm{d}r}{\mathrm{d}t}$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{(8+t^4)^2\times 336t^2 - 896t^6\times(8+t^4)}{(8+t^4)^4}$$

$$= \frac{2688t^2-560t^6}{(8+t^4)^3} \qquad \textit{(M1)(A2)}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2}=0 \Rightarrow 2688t^2-560t^6=0$$

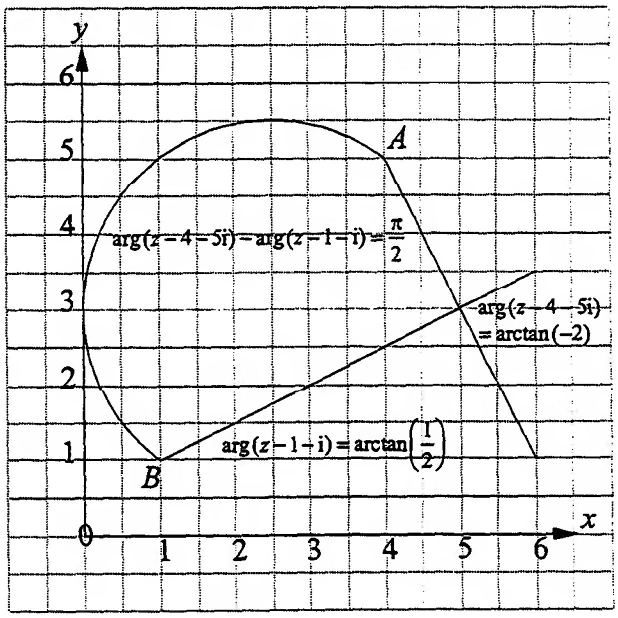
$$112t^2(24-5t^4)=0$$

$$t=0 \text{ or } t=\sqrt[4]{\frac{24}{5}} \text{ or } 1.48 \qquad \textit{(A1)}$$

But $t>0$ and at $t=\sqrt[4]{\frac{24}{5}}$, $\frac{\mathrm{d}r}{\mathrm{d}t}\neq 0$

Therefore the point of inflection occurs where $t=\sqrt[4]{\frac{24}{5}}$ or 1.48 *(R1)*

(ii) (a)



Axes drawn correctly *(A1)*
Points A and B plotted correctly *(A1)*

continued . . .

Question 2 continued

(b) (i) $\arg(z-4-5\mathrm{i}) = \arctan(-2)$

$\arctan\left(\dfrac{y-5}{x-4}\right) = \arctan(-2)$

OR $\arg(z-4-5\mathrm{i}) = \arctan(-2)$ is the line through $4+5\mathrm{i}$ with gradient $= -2$

$\dfrac{(y-5)}{(x-4)} = -2, \quad x>0, y>0$

$y = -2x+13$ *(M1)(A1)* (Plus line drawn correctly)

(ii) $\arg(z-1-\mathrm{i}) = \arctan\left(\dfrac{1}{2}\right)$

$\arctan\left(\dfrac{y-1}{x-1}\right) = \arctan\left(\dfrac{1}{2}\right)$

OR $\arg(z-1-\mathrm{i}) = \arctan\left(\dfrac{1}{2}\right)$ is the line through $1+\mathrm{i}$ with gradient $= \dfrac{1}{2}$

$\dfrac{y-1}{x-1} = \dfrac{1}{2}, \quad x>0, y>0$

$y = \dfrac{1}{2}x + \dfrac{1}{2}$ *(M1)(A1)* (Plus line drawn correctly)

(c) The gradient of $y=-2x+11$ is $-2 = m_1$

The gradient of $y = \dfrac{1}{2}x+\dfrac{1}{2}$ is $\dfrac{1}{2} = m_2$

Therefore $m_1 m_2 = -1$

Therefore $\theta = \dfrac{\pi}{2}$ *(M1)(A1)*

(d) Let $\arg(z-4-5\mathrm{i}) = \theta_1$ and $\arg(z-1-\mathrm{i}) = \theta_2$

then $\arg(z-4-5\mathrm{i}) - \arg(z-1-\mathrm{i}) = \dfrac{\pi}{2}$

becomes $\theta_1 - \theta_2 = \dfrac{\pi}{2}$, where $\theta_1 - \theta_2$ is the angle, at the point of intersection, between a line from A and a line from B. *(R2)*

Method 1

The locus is the set of points of intersection of a line from A and a line from B, such that $\theta_1 - \theta_2 = \dfrac{\pi}{2}$.

This is the semi-circle with AB as diameter.

That is semi-circle centre $\dfrac{5}{2}+3\mathrm{i}$, radius $\dfrac{5}{2}$ units. *(R1)(A2)* (Plus semi-circle correctly sketched)

continued . . .

Question 2 continued

Method 2

$$\theta_1 - \theta_2 = \frac{\pi}{2}$$

$$\Leftrightarrow \tan(\theta_1 - \theta_2) = \tan\frac{\pi}{2}$$

$$\Leftrightarrow \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \tan\frac{\pi}{2}$$

$$\Leftrightarrow 1 + \tan\theta_1 \tan\theta_2 = 0$$

$$\Leftrightarrow 1 + \frac{(y-5)}{(x-4)} \times \frac{(y-1)}{(x-1)} = 0$$

$$\Leftrightarrow x^2 + y^2 - 5x - 6y + 9 = 0$$ *(M1)(A1)*

This is the circle centre $\frac{5}{2} + 3\mathrm{i}$, radius $\frac{5}{2}$ units.

To satisfy the condition $\theta_1 - \theta_2 = \frac{\pi}{2}$, the locus is only the <u>semi-circle</u> shown. *(R1)(A2)* (Plus semi-circle correctly sketched)

3. (a) (i) Let $\vec{n}_1 = 3\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{n}_2 = -2\vec{i} + \vec{j} - 5\vec{k}$
be the normal vectors to the planes P_1 and P_2 respectively.

Then $\vec{n}_1 \times \vec{n}_2$ is parallel to the line of intersection, L, of the two planes. *(R1)*

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ -2 & 1 & -5 \end{vmatrix} = 3\vec{i} + 11\vec{j} + \vec{k}$$ *(M2)*

(ii) The point A is on $L \Leftrightarrow$ it is on each plane. *(R1)*

Let $\vec{a} = -\vec{j} - \vec{k}$, be the position vector for A.

Then $\vec{a} \cdot \vec{n}_1 = (3 \times 0) + (-1 \times -1) + (2 \times -1) = -1$, therefore A is on P_1
and $\vec{a} \cdot \vec{n}_2 = (-2 \times 0) + (1 \times -1) + (-5 \times -1) = 4$, therefore A is on P_2. *(M2)*

Therefore the equation of L is $\underline{r} = -\vec{j} - \vec{k} + \lambda(3\vec{i} + 11\vec{j} + \vec{k})$ *(A1)* Any equivalent form accepted

Note: candidates may use the cartesian equations to show that point A lies in each plane. Full marks should be awarded for this method.

(b) To find where the planes intersect we find the solution of the equivalent matrix equation:

$$\begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & -5 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ c \end{pmatrix}$$

This gives the augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1 \\ -2 & 1 & -5 & 4 \\ -4 & 1 & 1 & c \end{array}\right)$$ *(M1)*

Using row operations we obtain:

$$\begin{array}{r} \\ 3r_2 + 2r_1 \\ r_3 - 2r_2 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1 \\ 0 & 1 & -11 & 10 \\ 0 & -1 & 11 & c-8 \end{array}\right)$$

$$\begin{array}{r} \\ \\ r_2 + r_3 \end{array}\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1 \\ 0 & 1 & -11 & 10 \\ 0 & 0 & 0 & c+2 \end{array}\right)$$

Other methods of elimination possible

continued . . .

Question 3 continued

This is equivalent to the matrix equation

$$\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & -11 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ c+2 \end{pmatrix}$$ *(M2)*

The third row gives $0 = c + 2$

Therefore a solution is possible, that is the planes intersect, if $c = -2$.   *(A1)*

In this case we can find a solution for any value of z. Therefore there must be a line of intersection.   *(R1)*

(c) (i) P_3 is parallel to L if, $\vec{n}_3 = -4\vec{i} + \vec{j} + \vec{k}$, the normal vector to P_3, is orthogonal to, $\vec{d} = 3\vec{i} + 11\vec{j} + \vec{k}$, a direction vector for L.

$\vec{n}_3$ and $\vec{d}$ are orthogonal if their dot product is zero.   *(R1)*

$\vec{n}_3 \cdot \vec{d} = (-4 \times 3) + (1 \times 11) + (1 \times 1) = 0$ *(M1)*

Therefore P_3 is parallel to L.

(ii) The required distance, s, is equal to the distance between P_3 and point A since L contains A and is parallel to P_3.   *(R1)*

This distance is given by $s = \left| \vec{a} \cdot \hat{\vec{n}}_3 - \vec{r} \cdot \hat{\vec{n}}_3 \right|$

where $\vec{a}$ is the position vector for A
$\hat{\vec{n}}_3$ is the unit normal to the plane P_3 and
$\vec{r} \cdot \hat{\vec{n}}_3$ is the distance of P_3 from the origin.

$$s = \left| \frac{(0 \times 4) + (-1 \times 1) + (-1 \times 1)}{3\sqrt{2}} - \frac{5}{3\sqrt{2}} \right|$$

$$s = \left| \frac{-2}{3\sqrt{2}} - \frac{5}{3\sqrt{2}} \right|$$

$$s = \frac{7}{3\sqrt{2}} \text{ or } \frac{7\sqrt{2}}{6} \text{ units}$$ *(M1)(A2)*

4. (i) Let B be the event "the girl travels by bus"
Let TR be the event "the girl travels by train"
Let TA be the event "the girl travels by taxi"
Let L be the event "the girl is late for school"
Let NL be the event "the girl is not late for school"

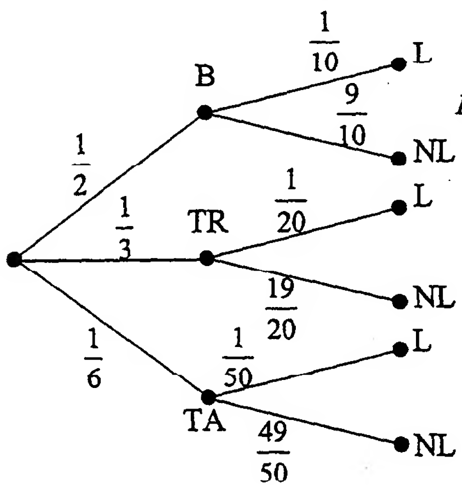
Then $p(B) = \frac{1}{2}$, $p(TR) = \frac{1}{3}$, $p(TA) = \frac{1}{6}$,
$p(L|B) = \frac{1}{10}$, $p(L|TR) = \frac{1}{20}$, $p(L|TA) = \frac{1}{50}$

(a)
$$\begin{aligned} p(L) &= p(B)\cdot p(L|B) + p(TR)\cdot p(L|TR) + p(TA)\cdot p(L|TA) \\ &= \left(\frac{1}{2}\times\frac{1}{10}\right)+\left(\frac{1}{3}\times\frac{1}{20}\right)+\left(\frac{1}{6}\times\frac{1}{50}\right) \\ &= \frac{1}{20}+\frac{1}{60}+\frac{1}{300} \\ &= \frac{7}{100} \end{aligned}$$

(M2)(A1) (or equivalent)

OR

Students may do this question using the tree diagram below



$p(B\cap L) = \frac{1}{20}$

$p(TR\cap L) = \frac{1}{60}$

$p(TA\cap L) = \frac{1}{300}$

$$\begin{aligned} p(L) &= p(B\cap L) + p(TR\cap L) + p(TA\cap L) \\ &= \frac{7}{100} \end{aligned}$$

(M2)(A1) (or equivalent)

continued . . .

Question 4 continued

(b) The required probability is $p(B|L)$ *(C1)*

EITHER

$$p(B|L) = \frac{p(B\cap L)}{p(L)} = \frac{\frac{1}{2}\times\frac{1}{10}}{\left(\frac{7}{100}\right)} = \frac{5}{7}$$

(M1)(A1) (or equivalent)

OR

From the tree diagram

$$p(B|L) = \frac{p(B\cap L)}{p(B\cap L)+p(TR\cap L)+p(TA\cap L)} = \frac{5}{7}$$

(M1)(A1) (or equivalent)

(c) Let X be the discrete random variable, "the cost, paid in dollars, for a journey to school".

X can take the values \$ 0.50, \$1.80 or \$9.00.

The probability distribution for X is

Cost (x)	\$0.50	\$1.80	\$9.00
$p(X=x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$E(X) = \sum_{\text{all } x} x\times p(X=x) = 0.25+0.60+1.50 = 2.35$$

(M2)(A1)

Therefore the expected cost for 180 journeys is \$ 423. *(A1)*

continued . . .

Question 4 continued

4. (ii) Let X be the random variable, "the number of people who book a seat on the airplane and show up for the flight".

Let p be the probability that a person who books a seat shows up for the flight.

Let q be the probability that a person who books a seat does not show up for the flight.

Let n be the number of people who book a seat.

Therefore $p = 0.92$ and $q = 1 - p = 0.08$

Then $X \sim \text{Bin}(317, 0.92)$

and $E(X) = np = 317 \times 0.92 = 291.64$ *(M1)(A1)*

and $\text{Var}(X) = npq = 23.3312$

$\sigma = \sqrt{npq} \approx 4.83024$ *(A1)*

For large n, and p not too small or too large,

$X \sim N(np, npq)$

We want to know $p(300 < X \leq 317)$ using $N(291.64, 23.3312)$

Since we are using a continuous distribution to approximate a discrete variable, we need to make a continuity correction.

Therefore, $p(300 < X \leq 317)$ transforms to $p(300.5 < X < 317.5)$. *(M1)*

Therefore $p(300.5 < X < 317.5)$

$$= p\left(\frac{300.5 - 291.64}{4.830} < \frac{X - 291.64}{4.830} < \frac{317.5 - 291.64}{4.830}\right)$$

$$= p(1.834 < Z < 5.354)$$ *(M1)(A1)*

$$= 1 - \left[0.9664 + \left(\frac{4}{10} \times 0.0007\right)\right]$$

$$= 1 - 0.9667$$

$$= 0.0333$$ (May be obtained from calculator)

$$= 3.33\%$$

Therefore the percentage of flights that are over booked is 3.33%. *(M1)(A1)*

5. (i) (a) (i)

$\bullet$	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	D	C	E	I	B
B	B	E	I	D	C	A
C	C	B	A	I	E	D
D	D	I	E	B	A	C
E	E	C	D	A	B	I

(A4)

1 or 2 errors deduct 1 mark
3 or 4 errors deduct 2 marks
5 or 6 errors deduct 3 marks
7 or more errors deduct 4 marks

(ii) $(T, \bullet)$ is closed
Matrix multiplication of 3×3 matrices is associative
I is the identity element
A and D are inverses of each other and B, C and E are self-inverse *(C4)*

Therefore $(T, \bullet)$ is a group

(iii)

Element	Order
I	1
A	3
B	2
C	2
D	3
E	2

$A^2 = D$
$A^3 = I \Rightarrow A$ is order 3 and D is order 3
$B^2 = I$
$C^2 = I$
$E^2 = I \Rightarrow B, C, E$ are order 2

(M1)(A2) Deduct 1 mark for each error

(b) Rearranging the table for $(T, \bullet)$ and comparing with that for $(U, *)$

$\bullet$	I	A	D	B	C	E
I	I	A	D	B	C	E
A	A	D	I	C	E	B
D	D	I	A	E	B	C
B	B	E	C	I	D	A
C	C	B	E	A	I	D
E	E	C	B	B	A	I

$*$	I	R_1	R_2	L	M	N
I	I	R_1	R_2	L	M	N
R_1	R_1	R_2	I	M	N	L
R_2	R_2	I	R_1	N	L	M
L	L	N	M	I	R_2	R_1
M	M	L	N	R_1	I	R_2
N	N	M	L	R_2	R_1	I

(A3)

Since the tables have exactly the same structure the two groups are isomorphic. *(R1)*

The mapping $f: T \mapsto U$ is defined as

$$f(I) \mapsto I$$
$$f(A) \mapsto R_1$$
$$f(B) \mapsto L$$
$$f(C) \mapsto M$$
$$f(D) \mapsto R_2$$
$$f(E) \mapsto N$$

(A2)

This is one-to-one and onto and is operation preserving,
that is $f(x \bullet y) = f(x) * f(y)$ for all $x, y \in T$ *continued . . .*

NOTE: other isomorphisms are possible.

Question 5 continued

(ii) (a) (i) $r \odot r = p$ Since p is the identity element this suggests that r is order 2. *(A1)(R1)*

If $(S, \odot)$ is a group then the order of r must be a factor of 5.
2 is not a factor of 5, therefore $(S, \odot)$ is not a group. *(R1)*

(ii) $q \odot (t \odot s) = q \odot p = q$
but $(q \odot t) \odot s = p \odot s = s$ *(A2)*
$q \odot (t \odot s) \neq (q \odot t) \odot s$ therefore $\odot$ is not associative on S. *(R1)*

(iii)

Element x	Element y
p	p
q	t
r	r
s	q
t	s

(A1)

Axiom If $(S, \odot)$ is a group then the inverse of an element x is x^{-1} such that $x \odot x^{-1} = x^{-1} \odot x = p$ (identity element in S) *(C1)*

From the above $q \odot t = p \Rightarrow t = q^{-1}$
but $t \odot q = s$

Therefore the axiom stated above is not satisfied. *(R2)*

(ii) (b) (i)

#	p	q	r	s	t
p	p	q	r	s	t
q	q	r	t	p	s
r	r	t	s	q	p
s	s	p	q	t	r
t	t	s	p	r	q

(A3)

1 or 2 errors deduct 1 mark
3 or 4 errors deduct 2 marks
5 or more errors deduct 3 marks

$(S, \#)$ is closed
is associative on S (given)
p is the identity element
q and s are inverses of each other
r and t are inverses of each other *(C3)*

Therefore $(S, \#)$ is a group.

continued . . .

Question 5 continued

(ii) A group of order n is cyclic $\Leftrightarrow$ it contains an element of order n. *(R1)*

METHOD 1

$q^2 = r$	$r^2 = s$	
$q^3 = t$	$r^3 = q$	
$q^4 = s$	$r^4 = t$	
$q^5 = p \Rightarrow q, s$ are order 5	$r^5 = p \Rightarrow r, t$ are order 5	*(M1)*

Therefore $(S, \#)$ is cyclic and q, r, s, t are generators *(A1)*

METHOD 2

If $(S, \#)$ is a group the order of each element is a factor of 5
Therefore q, r, s, t must be order 5 *(R1)*

Therefore $(S, \#)$ is cyclic and q, r, s, t are generators *(A1)*

(ii) (c) Let $(G, \circ)$ be a finite cyclic group, with generator $z \in G$.
Let $x, y \in G$. Then there are positive integers m and n such that

$$x = z^m$$
$$y = z^n$$ *(R1)*

Therefore $x \circ y = z^m \circ z^n = z^{m+n} = z^{n+m} = z^n \circ z^m = y \circ x$ *(M2)*

Therefore $(G, \circ)$ is Abelian *(R1)*

6. (i) (a)

Graph	f	e	v	k
G_1	5	9	6	$\frac{5}{9}$
G_2	6	12	8	$\frac{1}{2}$
G_3	7	15	10	$\frac{7}{15}$

(A2) Deduct 1 mark for each error

(b) Let G be a connected planar graph with e edges, v vertices and f faces in a planar representation of G. Then Euler's formula is $f+v-e=2$.
(C1)

For $G_1 : f+v-e=5+6-9=2$
For $G_2 : f+v-e=6+8-12=2$
For $G_3 : f+v-e=7+10-15=2$

Therefore G_1, G_2 and G_3 satisfy Euler's formula. *(M1)*

(c) The degree of a face is the number of edges on the boundary of the face. If G is a connected planar simple graph with $v \geq 3$, then the degree of each face ≥ 3. The sum of the number of edges bounding each face is at most $2e$.
(R2)

Therefore $2e \geq$ the sum of degrees of the faces $\geq 3f$
Therefore $2e \geq 3f$
Therefore $\frac{2}{3} \geq \frac{f}{e}$
Therefore $k \leq \frac{2}{3}$
(M2)

(d) By 6(i)(b) Euler's formula gives $f+v-e=2$. But by 6(i)(c) $f \leq \frac{2}{3}e$.

Therefore $2+e-v \leq \frac{2}{3}e$
Therefore $6+3e-3v \leq 2e$
Therefore $e \leq 3v-6$ *(M3)*

(e) Consider a connected planar simple graph G with $v \geq 3$. Suppose that $\deg(v_i) \geq 6$ for all vertices v_i. ($\deg(v_i)$ is the degree of vertex v_i, where v_i is the ith vertex, $i = 1$ to v) *(R1)*

By the Handshaking Lemma the sum of all $\deg(v_i)$ is $2e$. *(R1)*

Therefore $2e \geq 6v$
Therefore $e \geq 3v$ But by 6(i)(d) $e \leq 3v-6$
Therefore $3v-6 \geq 3v$ *(M1)*
which is a contradiction so $\deg(v_i) \leq 5$ for at least one vertex v_i. *(R1)*

continued . . .

Question 6 continued

(ii)(a) The graph G_4 is simple with $v = 6$.

$\deg(v_1) = \deg(v_2) = \deg(v_3) = \deg(v_4) = \deg(v_5) = 3, \deg(v_6) = 5$ *(A2)*

For $G_4, v \geq 3$ and $\deg(v_i) \geq \frac{1}{2}v$, for every vertex v_i *(R1)*

Therefore by Dirac's Theorem G_4 is Hamiltonian.

A Hamiltonian circuit would be $v_1v_2v_3v_4v_5v_6v_1$. *(A1)* (Other circuits possible)

(b)(i) The graph G_5 is simple with $v = 7$.

$\deg(v_1) = \deg(v_4) = 2, \deg(v_2) = \deg(v_3) = \deg(v_5) = \deg(v_6) = \deg(v_7) = 4$ *(A2)*

G_5 is Hamiltonian with a Hamiltonian circuit $v_1v_2v_3v_4v_5v_7v_6v_1$. *(A2)* (Other circuits possible)

(Dirac's theorem does not apply in this case since the condition $\deg(v_i) \geq \frac{1}{2}v$ is not satisfied.)

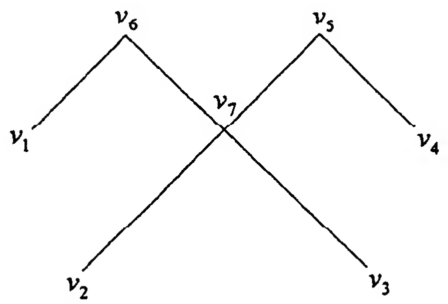
(ii) The graph G_5 is a connected simple graph. All $\deg(v_i)$ are even, therefore G_5 is Eulerian. *(C1)*

An Eulerian circuit would be $v_1v_2v_3v_4v_5v_6v_7v_5v_3v_7v_2v_6v_1$. *(A2)* (Other circuits possible)

(iii) A tree is a connected graph G which contains no circuits (cycles). *(C1)*

Let G be a connected graph. A spanning tree in G is a subgraph of G that includes all the vertices of G and is a tree. *(C1)*

A spanning tree for G_5 with degree sequence (1 , 1 , 1 , 1 , 2 , 2 , 4) would be



(A2) (Other answers possible)

continued . . .

(iii) METHOD 1

Using the Shortest Path Algorithm (Dijkstra's Algorithm):

	Permanent labels
$L(A) = 0$	
$L(B) = 2.1;\quad L(B) = 2.0 + 0.5 = 2.5$	$L(B) = 2.1$
$L(J) = 2.0;\quad L(J) = 2.1 + 0.5 = 2.6$	$L(J) = 2.0$
$L(C) = L(B) + 3.1 = 5.2$	$L(C) = 5.2$
$L(I) = L(J) + 1.8 = 3.8$	$L(I) = 3.8$
$L(D) = L(C) + 3.5 = 8.7;\quad L(D) = L(B) + 6.9 = 9.0$	
$L(D) = L(J) + 7.1 = 9.1;\quad L(D) = L(H) + 4.9 = 9.3$	$L(D) = 8.7$
$L(H) = L(I) + 1.6 = 5.4;\quad L(H) = L(J) + 5.5 = 7.5$	
$L(H) = L(B) + 7.8 = 9.9;\quad L(H) = L(D) + 4.9 = 13.6$	$L(H) = 5.4$
$L(G) = L(H) + 2.9 = 8.3$	$L(G) = 8.3$
$L(F) = L(H) + 7.5 = 12.9\quad L(F) = L(G) + 4.8 = 13.1$	
$L(F) = L(J) + 11.4 = 13.4;\quad L(F) = L(D) + 7.3 = 16.0$	
$L(F) = L(B) + 12.0 = 14.1$	$L(F) = 12.9$
$L(E) = L(D) + 8.0 = 16.7;\quad L(E) = L(F) + 1.5 = 14.4$	$L(E) = 14.4$

Deduct 1 mark for each error

(M4)(A4)

So quickest path is *A-J-I-H-F-E* which takes 14.4 minutes. *(A2)*

METHOD 2

In tabular form:

Deduct 1 mark for each error

	A	*J*	*B*	*I*	*C*	*H*	*G*	*D*	*F*	*E*
A	0	2.0	2.1							
J		2.0	2.1	3.8		7.5		9.1	13.4	
B			2.1		5.2	7.5		9.0	13.4	
I				3.8		5.4				
C					5.2			8.7		
H						5.4	8.3	8.7	12.9	
G							8.3		12.9	
D						5.4		8.7	12.9	16.7
F									12.9	14.4
E										14.4

(M4)(A4)

So quickest path is *A-J-I-H-F-E* which takes 14.4 minutes. *(A2)*

7. (i) (a) If $X_1, X_2 \ldots X_n$ is a random sample of size n drawn from a normally distributed population with mean μ and variance σ^2 then the sample mean $\overline{X}$ is normally distributed with $E(\overline{X}) = \mu$ and $\text{Var}(\overline{X}) = \dfrac{\sigma^2}{n}$. *(C2)*

(b) A sample of 10 boxes was taken from a normally distributed population of boxes, $N \sim (1.006, 0.003)$.

Set up the following hypotheses:

H_0: settings are unaltered and $\mu = 1.006$.

H_1: settings are altered and $\mu \neq 1.006$. *(C2)*

It is necessary to perform a two-tail test since we are asked whether the mean has changed (it could be greater or less than 1.006 kg). Since the population standard deviation is known (0.003) a z-test is appropriate. ***(R2)***

For the given sample $\bar{x} = 1.0036$. *(A1)*

Therefore $z = \dfrac{|\bar{x} - \mu|}{\frac{\sigma}{\sqrt{n}}} = \dfrac{|1.0036 - 1.006|}{\frac{0.003}{\sqrt{10}}} = 2.53$ *(M1)(A1)*

From the tables at 5% level the critical value of $z = 1.96$. *(A1)*

The calculated value for $z > 1.96$ therefore it is significant at the 5% level and we reject H_0.

We conclude that the settings have changed and $\mu \neq 1.006$. ***(R2)***

(c) (i) Based on this sample a 95% confidence interval for μ is given by:

$$\left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \quad , \quad \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right)$$
$$\left(1.0036 - 1.96\frac{0.003}{\sqrt{10}}, 1.0036 + 1.96\frac{0.003}{\sqrt{10}}\right)$$

The 95% confidence interval for μ is $(1.002\ ,\ 1.005)$, correct to 3 decimal places. *(M1)(A2)*

continued...

continued...

Question 7 continued

(ii) If we assume that $\mu \neq 1.006$ and that the correct mean is 1.0036, then the probability that a box is under weight is given by

$$p(\bar{X} < 1) = p\left(Z < \frac{1 - 1.0036}{0.003}\right) \qquad \textit{(M1)}$$

$$= p(Z < (-1.2))$$
$$= 1 - \Phi(1.2)$$
$$= 1 - 0.8849$$
$$= 0.1151 \qquad \textit{(M1)(A1)}$$

Hence for a delivery of 10000 boxes approximately 1151 are underweight. *(A1)* (Accept 1150)

(ii) Set up the following hypotheses:

H_0: There is an equal number of each colour of ball.
H_1: There is not an equal number of each colour of ball. *(C2)*

If there is an equal number of each colour of ball we would expect each colour of ball to be drawn an equal number of times, i.e. 40.

This gives the table:

Colour	Blue	Red	Green	Yellow	Black	
Observed Frequency (O)	38	49	35	34	44	Total 200
Expected Frequency (E)	40	40	40	40	40	Total 200

(A1)

Then, $\nu = 5 - 1 = 4$ degrees of freedom, and we consider the $\chi^2(4)$ distribution. *(A1)*

We will test at the 10% level and reject H_0 if $\chi^2 > \chi^2_{10\%}(4)$ i.e. if $\chi^2 > 7.78$. *(R1)*

$$\chi^2 = \sum \frac{(O-E)^2}{E} = \frac{4}{40} + \frac{81}{40} + \frac{25}{40} + \frac{36}{40} + \frac{16}{40} = 4.05 \qquad \textit{(M1)(A1)}$$

Since $\chi^2 < 7.78$ we do not reject H_0 and conclude that there is an equal number of each colour of ball in the bag. *(R2)*

continued...

Question 7 continued

(iii) Set up the following hypotheses:

H_0: The course had no effect on test scores. The distribution of differences in (paired) scores has a mean of zero, i.e. $\mu = 0$

H_1: The course improved the test scores. The distribution of differences in (paired) scores has a mean greater than zero, i.e. $\mu > 0$ *(C2)*

The differences d_i are given by "Test Score at end of course" – "Test Score before course".

$d_1 = 8; d_2 = 1; d_3 = 2; d_4 = -2; d_5 = 6.$ *(A1)*

$\bar{d} = 3$ and $s = 4$ (where s is the unbiased estimate of the population standard deviation) *(A1)(A1)*

Since the population mean is unknown, and we are testing if $\mu > 0$, we use a one-tail t test with 4 degrees of freedom. *(R2)*

$$t_4 = \left(\frac{\bar{x} - \mu}{\frac{s}{\sqrt{5}}} \right)$$

$$= \left(\frac{3 - 0}{\frac{4}{\sqrt{5}}} \right) = 1.68$$

(M1)(A1)

At 5% confidence level and 4 degrees of freedom the critical value of $t_4 = 2.132$ *(A1)*

Since the calculated value of $t < 2.132$ there is no evidence to reject H_0. Therefore we conclude the course had no significant effect on the scores. *(R2)*

8. (i) (a) The trapezium rule estimate T_n of the definite integral $\int_a^b f(x)\mathrm{d}x$ where the interval, $a \le x \le b$, is divided into n equal subintervals of width h is:

$$T_n = \frac{h}{2}[f(a) + 2f(a+h) + 2f(a+2h) + \ldots + 2f(a+(n-1)h) + f(b)]$$

where $h = \dfrac{b-a}{n}$ *(C3)*

(b) $\int_a^b f(x)\mathrm{d}x = T_n + E_n$, where E_n the error term is $E_n = -\dfrac{(b-a)h^2}{12} f''(c)$ where c is between a and b. *(C2)*

If $f(x) = Ax + B$ then $f''(x) = 0$, for all values of x.

Therefore $E_n = 0$ and the trapezium rule is exact. *(M1)(R1)*

(c) Applying the trapezium rule to the integral $\int_1^{2n+1} x\,\mathrm{d}x$ with $h = 2$ gives

$$T_n = \int_1^{2n+1} x\,\mathrm{d}x = \frac{2}{2}[f(1) + 2f(3) + 2f(5) + \ldots + 2f(2n-1) + f(2n+1)] \quad \textit{(M1)(A1)}$$

$$= 1 + 2[3 + 5 + \ldots + (2n-1)] + (2n+1)$$

$$= 2[1 + 3 + 5 + \ldots + (2n+1)] - 1 - (2n+1). \quad \textit{(M1)(A1)}$$

Hence

$$1 + 3 + 5 + \ldots + (2n+1) = \frac{1}{2}\int_1^{2n+1} x\,\mathrm{d}x + \frac{1}{2}(1 + (2n+1)) \quad \textit{(M1)(A1)}$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_1^{2n+1} + n + 1$$

$$= \frac{(2n+1)^2 - 1}{4} + n + 1 \quad \textit{(M1)(A1)}$$

$$= \frac{4n^2 + 4n + 1 - 1 + 4n + 4}{4}$$

$$= (n+1)^2. \quad \textit{(M1)(A1)}$$

continued...

Question 8 continued

(ii)(a) For the series $\sum_{k=1}^{\infty}(-1)^k \frac{2k}{4k-3}$,

$$\lim_{k\to\infty}\frac{2k}{4k-3} = \lim_{k\to\infty}\frac{2}{4-\frac{3}{k}} = \frac{2}{4} = \frac{1}{2} \neq 0$$ *(M2)(A1)*

Therefore by the Divergent Series Test, the series $\sum_{k=1}^{\infty}(-1)^k \frac{2k}{4k-3}$ diverges. *(R1)*

(b) For the series $\sum_{k=1}^{\infty}\frac{1}{3k^2-2k}$,

Consider $3k^2-2k = k^2+2k^2-2k = k^2+2k(k-1) \geq k^2$

Therefore $\frac{1}{3k^2-2k} \leq \frac{1}{k^2}$ *(M2)(R1)*

But $\sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^p$ converges for $p>1$, therefore the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges and by the Comparison Test, the series $\sum_{k=1}^{\infty}\frac{1}{3k^2-2k}$ also converges. *(M1)(R1)*

(c) For the series $\sum_{k=1}^{\infty}\frac{(k+1)!}{(k+1)^3}$, put $u_k = \frac{(k+1)!}{(k+1)^3}$.

Then $\frac{u_{k+1}}{u_k} = \frac{(k+2)!}{(k+2)^3}\times\frac{(k+1)^3}{(k+1)!} = \frac{(k+2)(k+1)^3}{(k+2)^3} = \frac{(k+1)^3}{(k+2)^2} > 1, \quad k = 2,3,\ldots$

(M2)(A2)

Therefore by the Ratio Test the series $\sum_{k=1}^{\infty}\frac{(k+1)!}{(k+1)^3}$ diverges. *(R2)*

continued...

Question 8 continued

(iii) Writing the equation $x = \tan x$ as $x = \tan(x - \pi)$ and taking arctan of each side gives:

$$\arctan x = x - \pi \text{, since } x \text{ is inside the interval } \frac{\pi}{2} \le x \le \frac{3\pi}{2}.$$

Therefore $x = \pi + \arctan x$ *(A1)*

Let $g(x) = \pi + \arctan x$, then $g(x)$ is continuous on $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ and differentiable on $\frac{\pi}{2} < x < \frac{3\pi}{2}$, and $g'(x) = \frac{1}{1+x^2}$ and $0 < g'(x) < 1$ on $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ *(C3)*

Hence $x_{n+1} = \pi + \arctan x_n$ will give a convergent sequence of $x_1, x_2, x_3 \cdots$ *(R1)*

Taking $x_1 = 4.5$ gives:

$$x_2 = \pi + \arctan(4.5) = 4.493720035$$
$$x_3 = 4.493424113$$
$$x_4 = 4.493410149$$
$$x_5 = 4.493409491$$

This gives the solution $x = 4.493409$ correct to six decimal places. *(M1)(A2)*